The Timing of Uncertainty

and

The Intensity of Policy

by

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Abstract

This article analyzes the trade-off between "caution" and "intensity" in the use of the control variable in a one-state one-control dynamic stochastic quadratic linear optimization problem with discount factor. It studies the effects that changes in uncertainty of the control parameter have on the optimal first-period response of the control variable, showing that the trade-off between "caution" and "intensity" depends on the timing of the uncertainty.

Given an increase in current uncertainty and an equal increase in future uncertainty, caution will always prevail over intensity. Moreover, the prevalence of caution will be enlarged as the increase in future uncertainty moves farther away into the future, while this prevalence will be reduced as the increase in future uncertainty expands into the future.

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1. Introduction

The effects of parameter uncertainty on optimal policy have received increasing attention over the last few years, particularly in the field of macroeconomic policy analysis. [1]Special attention has been paid to the fact that central banks tend to use their policy tools in a cautious way.[2]

Important theoretical results were developed in an earlier group of studies, building on the seminal Brainard (1967) paper which showed, for a static model, that an increase in uncertainty would result in a more cautious use of the policy variable. Brainard's results were extended to dynamic models by Chow (1973), Turnovsky (1975) and Shupp (1976) for the case of current uncertainty. Craine (1979) pointed out the relevance of the effect of future uncertainty, since this would induce a more aggressive -instead of a more cautious- use of the policy variable, a less familiar effect than the caution effect.[3] These studies dealt mostly with one-state one-control models.

Since current uncertainty induces "caution" while future uncertainty induces "intensity", it is natural to ask what happens when both types of uncertainty are present. This article analyzes the trade-off between caution and intensity in the first-period response of the control variable when there is a change in both current and future uncertainty.[4] It focuses on a one-state one-control dynamic stochastic quadratic linear optimization problem, and it uses the Riccati equations to study the dynamic link between future uncertainty and first-period policy response. The findings are that given an increase in current uncertainty and an equal increase in future uncertainty of the control parameter, caution will always prevail over intensity. The prevalence of caution will be enlarged as the increase in future uncertainty *moves* farther away into the

[1] See for example Wieland (1998), Amman and Kendrick (1999), Mercado and Kendrick (1999 and 2000) and Sack (2000). The analysis of optimal policy under uncertainty can be approached in different ways besides the one used here, focused on optimal control under parameter uncertainty. For a road map of approaches, see Christodoulakis et. al (1993).

[2] For a recent survey on this issue see Clarida et. al (1999).

[3] Mercado and Kendrick (2000) extend this result to a one-state two-control model, showing that given an increase in future uncertainty there will be an increase use of both, of at least one, of the first-period controls, depending on the relative magnitude of their first-period weighted variances.

[4] It is usual to focus on the first-period behavior of the policy variable, since its qualitative behavior may well change beyond the first period. This is because its optimal values after the first period will be

future (from a given future period to a more distant one), while it will be reduced as the increase in future uncertainty *expands* into the future, that is, to several future periods.

2. The Optimization Problem

Consider a one-state one-control Dynamic Stochastic Quadratic Linear Discounted Problem, where the control parameter is uncertain. Formally, the problem is expressed as one of finding the controls $(u_k)_{k=0}^{N-1}$ to minimize a quadratic criterion function J of the form:

$$J = E\left\{\frac{1}{2}\boldsymbol{\beta}^N w x_N^2 + \frac{1}{2}\sum_{k=0}^{N-1}\boldsymbol{\beta}^k\left(w x_k^2 + \lambda u_k^2\right)\right\} \tag{2.1}$$

subject to:

$$x_{k+1} = a x_k + b u_k + \varepsilon_k \tag{2.2}$$

where:

E = expectation operator
$\boldsymbol{\beta}$ = discount factor $(0 < \boldsymbol{\beta} \le 1)$
x = state variable
u = control variable
w = positive weight on the state variable
λ = positive weight on the control variable
a = state parameter
b = control parameter
σ_b^2 = parameter b variance
ε = random disturbance

Desired paths for the state and the controls are zero.[5] Parameter means a and b and variance σ_b^2 are assumed to be known, and σ_b^2 may vary over time. Finally, the absolute value of the state parameter a is assumed to be smaller or equal to one (the state equation is not unstable). The solution to the problem is the feedback rule (see Kendrick (1981), Ch. 6 and Amman et. al (1995), Appendix A):

computed recursively, using the model equation to determine the value of the state variable for period "$k+1$" (see Kendrick 1981, Chapter 6).

$$u_k = G_k x_k \tag{2.3}$$

where the feedback gain coefficient is:

$$G_k = -\left(E\left\{\lambda + \beta k_{k+1} b^2\right\}\right)^{-1} E\left\{\beta k_{k+1} ab\right\} \tag{2.4}$$

and where k is the Riccati matrix (in this case, a scalar). Since $E\{b^2\} = b^2 + \sigma_b^2$, while σ_b^2 is allowed to vary with time, we can write:

$$G_k = -\frac{\beta k_{k+1} ab}{\lambda + \beta k_{k+1}\left(b^2 + \sigma_{b(k)}^2\right)}. \tag{2.5}$$

The evolution of k is given by the Riccati equations (see Kendrick (1981), Ch. 6 and Amman et. al, Appendix A):

$$k_N = w \tag{2.6}$$

for the terminal period N and:

$$k_k = E\left\{w + \beta k_{k+1} a^2\right\} - \left(E\left\{\lambda + \beta k_{k+1} b^2\right\}\right)^{-1}\left(E\left\{\beta k_{k+1} ab\right\}\right)^2 \tag{2.7}$$

for any other period. Taking expectations in (2.7) and re-arranging, we obtain:

$$k_k = w + \beta k_{k+1} a^2 \left[1 - \frac{\beta k_{k+1} b^2}{\lambda + \beta k_{k+1}\left(b^2 + \sigma_{b(k)}^2\right)}\right]. \tag{2.8}$$

The Riccati equations are solved by backward integration starting from period *"N-1"*. Since w provides a sort of "price information" about the value of keeping the system in state x_N at time N, the Riccati equations transmit this information from the last period backward in time. In what follows, the Riccati equations will be the main tool to study the link between future

[5] This case is common when working with log-linearized models or models with variables expressed in

uncertainty and optimal first-period control response. Thus, it is convenient to analyze in detail some of their properties.

Notice first that k will always be positive. Since w is positive, from (2.6) k_N will be positive. Given that k_N is positive, from (2.8) k_{N-1} will also be positive, since the numerator in the term between brackets in (2.8) is always smaller than the denominator, making that term always positive. By the same reasoning, since k_{N-1} is positive, we obtain that k_{N-2} will be positive, and so on. We can conclude that k will always be positive.

From (2.8) we obtain:

$$\frac{\partial k_k}{\partial k_{k+1}} = \beta a^2 \left[1 - \frac{2\lambda\beta k_{k+1} b^2 + \beta^2 k_{k+1}^2 b^4 + \beta^2 k_{k+1}^2 b^2 \sigma_{b(k)}^2}{\left(\lambda + \beta k_{k+1}\left(b^2 + \sigma_{b(k)}^2\right)\right)^2} \right] > 0 \; and < 1. \qquad (2.9)$$

The numerator in the term between brackets will always be smaller than the denominator,[6] making that term always positive and smaller than one. Since $|a| \leq 1$ and $0 < \beta \leq 1$, (2.9) will always be positive and smaller than one. Finally, from (2.9) we obtain:

$$\frac{\partial^2 k_k}{\partial k_{k+1}^2} = -\frac{2\beta^2 \lambda^2 a^2 b^2 \left(\beta k_{k+1} b^2 + \lambda + \beta k_{k+1} \sigma_{b(k)}^2\right)}{\left(\lambda + \beta k_{k+1}\left(b^2 + \sigma_{b(k)}^2\right)\right)^4} < 0 \qquad (2.10)$$

since λ, k and β are positive. It follows that (2.8) is a strictly concave function. The information provided by (2.6), (2.9) and (2.10) is represented in the phase diagram below, which characterizes the dynamics of the Riccati equations, where "ss" means steady-state.

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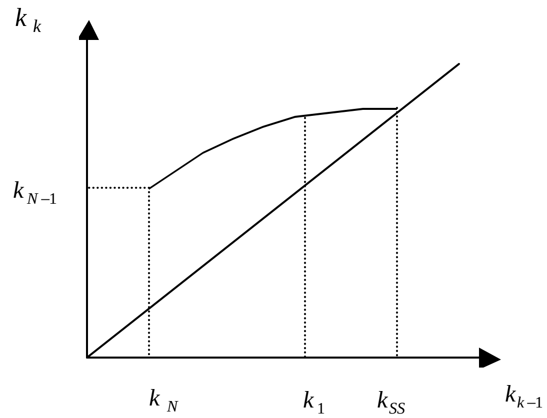
percentage changes with respect to a base case.

[6] Since the expanded denominator:

$$2\lambda\beta k_{k+1} b^2 + \beta^2 k_{k+1}^2 b^4 + 2\beta^2 k_{k+1}^2 b^2 \sigma_{b(k)}^2 + \lambda^2 + 2\lambda\beta k_{k+1} \sigma_{b(k)}^2 + \beta^2 k_{k+1}^2 \sigma_{b(k)}^4$$

contains all the numerator's terms.



From the diagram it follows that k will always be positive and increasing as we move backwards from the terminal period:

$$k_1 > k_2 \;\ldots > k_{N-1} > k_N \;\; . \tag{2.11}$$

3. Current Uncertainty, Future Uncertainty and Optimal First-Period Response

Here the interest is in the effects on the first-period response of the policy variable u (or equivalently, on $|G|$, the absolute value of G) when there is a increase in s_b^2, the uncertainty associated with the parameter which is multiplied by the control variable. Since l and k are positive, and given that the absolute value of quadratic terms is their own value, from (2.5) we can write :

$$\left|G_k\right| = \frac{\mathbf{b}\, k_{k+1}\left|ab\right|}{l + \mathbf{b}\, k_{k+1}\left(b^2 + s_{b(k)}^2\right)} \; . \tag{3.1}$$

From (3.1) we obtain the effect on the first-period response in the policy variable when there is a increase in current uncertainty:

$$\frac{\partial |G_0|}{\partial \sigma^2_{b(0)}} = -\frac{\beta^2 k_1^2 |ab|}{\left(\lambda + \beta k_1 \left(b^2 + \sigma^2_{b(0)}\right)\right)^2} < 0 \; . \qquad (3.2)$$

This is the well known result that an increase in current uncertainty will always induce a more cautions first-period response from the control variable *u*.

Consider the case of an increase in future uncertainty $\sigma^2_{b(T)}$, where *T* can take any value between *1* and *(N-1)*. The multiperiod link between future uncertainty $\sigma^2_{b(T)}$ and the absolute value of the first-period feedback gain coefficient G_0 is made by the successive Riccati equations. From (3.1) and (2.8) we obtain:

$$\frac{\partial |G_0|}{\partial \sigma^2_{b(T)}} = \frac{\partial |G_0|}{\partial k_1} \quad \ldots \quad \frac{\partial k_k}{\partial k_{k+1}} \quad \ldots \quad \frac{\partial k_T}{\partial \sigma^2_{b(T)}} > 0 \qquad (3.3)$$

since also from (3.1) and (2.8) we obtain, respectively:

$$\frac{\partial |G_0|}{\partial k_1} = \frac{\lambda |ab| \beta}{\left(\lambda + \beta k_1 \left(b^2 + \sigma^2_{b(0)}\right)\right)^2} > 0 \qquad (3.4)$$

$$\frac{\partial k_T}{\partial \sigma^2_{b(T)}} = \frac{\beta^3 k_{T+1}^3 a^2 b^2}{\left(\lambda + \beta k_{T+1} \left(b^2 + \sigma^2_{b(T)}\right)\right)^2} > 0 \qquad (3.5)$$

while from (2.9) we know that $\frac{\partial k_k}{\partial k_{k+1}} > 0$. Consequently, an increase in future uncertainty will always induce a more intense first-period response from the control variable *u*. Facing an increase in future uncertainty, it makes sense to use more intensely the first-period control since it has, in relative terms, a more predictable impact on the state variable.[7] [8]

What will be the effect on the first-period response in the policy variable u when there is an equal increase in both current and future uncertainty? This amounts to compare the absolute value of (3.2) against (3.3). From (3.2)-(3.5), simplifying and re-arranging, we obtain:

$$\left\{\left|\frac{\partial|G_0|}{\partial\sigma^2_{b(0)}}\right| = k_1^2\right\} > \left\{\frac{\partial|G_0|}{\partial\sigma^2_{b(T)}} = k_{T+1}^2 a^2 \beta \;\;\ldots\;\; \frac{\partial k_k}{\partial k_{k+1}} \;\;\ldots\;\; \left[\frac{\lambda\beta k_{T+1} b^2}{\left(\lambda + \beta k_{T+1}\left(b^2 + \sigma^2_{b(T)}\right)\right)^2}\right]\right\} \quad (3.6)$$

Indeed, the term between brackets on the right hand side will always be smaller than one, since the denominator contains all the numerator's terms.[9] From (2.9), all the corresponding terms of the form $\frac{\partial k_k}{\partial k_{k+1}}$ will always be smaller than one.[10] Finally, a^2 and β will always be smaller or equal to one. Therefore, the inequality condition in (3.6) will be satisfied if:

$$k_1^2 > k_{T+1}^2 \quad (3.8)$$

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[7] Notice that if $\lambda = 0$, (3.4) and thus (3.3) will be zero, since k_{k+1} vanishes from (3.1). With no weight on the control variable the link between future uncertainty and first-period response is broken, and the optimal policy becomes a function of contemporaneous parameters only.

[8] If parameter a is allowed to be stochastic (that is, if $\sigma^2_a \neq 0$) but uncorrelated with b we can write:

$$\frac{\partial|G_0|}{\partial\sigma^2_{a(T)}} = \frac{\partial|G_0|}{\partial k_1} \;\;\ldots\;\; \frac{\partial k_k}{\partial k_{k+1}} \;\;\ldots\;\; \frac{\partial k_T}{\partial\sigma^2_{a(T)}} > 0$$

since $\frac{\partial|G_0|}{\partial k_1}$ remains the same as (3.4); $\frac{\partial k_k}{\partial k_{k+1}}$ in (2.9) is modified by adding the term "$+\beta\sigma^2_a$" to its right hand side, thus it will still be greater than zero but it may or may not be smaller than one; and finally $\frac{\partial k_T}{\partial\sigma^2_{a(T)}} = \beta k_{T+1}$ which is positive, since β and k are positive. Therefore when a is stochastic an increase in its future variance will bring about a more intense first period optimal response. Notice that σ^2_a will remain absent from (2.5) and thus from (3.1). Thus, an increase in the current variance of a will not affect the optimal policy response. Allowing a and b to be correlated does not yield, in general, unambiguous results. See Craine (1979) and Holly and Hughes-Hallett (1989), Ch. 4.

[9] See footnote 5 for an expression of the expanded denominator.

which, from (2.11), will always be the case. Thus, g*iven an equal increase in both current an future uncertainty, caution will always prevail over intensity in the first-period optimal response of the control variable u.*[11]

What will be the effect on the first-period response in the policy variable u as the timing of the increase in uncertainty changes? There are two interesting cases. Consider first the case of an increase in future uncertainty which *moves* farther away into the future, e.g. from period T to $T+1$. In this case, (3.3) will become smaller, since it will contain an extra term of the form (2.9), which is greater than zero and smaller than one, while the last term (3.5) will decrease. Indeed, from (3.5) and since λ, β and k are positive, we obtain:

$$\frac{\partial^2 k_T}{\partial \sigma^2_{b(T)} \partial k_{T+1}} = \frac{\beta^3 k^2_{T+1} a^2 b^2 \left(3\lambda + \beta k_{T+1}\left(b^2 + \sigma^2_{b(T)}\right)\right)}{\left(\lambda + \beta k_{T+1}\left(b^2 + \sigma^2_{b(T)}\right)\right)^3} > 0 \tag{3.9}$$

and, from (2.11), we know that as we move from T to $T+1$, k will become smaller and so will (3.5), since (3.9) is positive. Consequently, *given an increase in current uncertainty and an equal increase in future uncertainty, the prevalence of caution will be enlarged as the increase in future uncertainty moves farther away into the future, since the intensity it induces on the first-period response of the control variable will decrease while the cautionary component of the policy response will remain the same.*[12]

Consider now the case in which the future increase in uncertainty *expands* from T into the future, that is, when it includes time periods from T to $T+m$, where $1 \le m \le N - T$. In this case the overall effect on $|G_0|$ will be the sum of $m+1$ effects, each of them computed as in the case of "moving" uncertainty analyzed above. We know that each of those effects will be positive and decreasing as m increases. Thus, *given an increase in current uncertainty and*

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[10] This may not be true if we allow parameter a to be stochastic, since $\frac{\partial k_k}{\partial k_{k+1}}$ may not always be smaller than one. See footnote 8.

[11] Notice that this result holds even in the case of no time discounting in the criterion function of the optimization problem, that is, when β=1.

[12] Indeed, notice that (3.2) is not affected by changes in future uncertainty.

an equal increase in future uncertainty, the prevalence of caution will be reduced as the increase in future uncertainty expands into the future. This expansion induces a growing intensity (though at a decreasing rate) in the first-period response of the control, while the cautionary component of the policy response will remain the same.[13]

A numerical example will help to illustrate these findings. Consider a problem with the following parameters: β = 1, w = 1, λ = 0.5, a = 0.9, b = -0.2, N = 5, and with an initial condition x_0 = 100. For a baseline solution with σ_b^2 = 0.001, the corresponding first-period optimal response is $u_{0(base)}$ = 79.88. Table 1 below shows the results of five experiments in which $\sigma_{b(T)}^2$ increases to 0.1.[14]

Table 1

T	u_0	$\lvert u_0 - u_{0(base)} \rvert$
0	55.50	24.38
1	82.32	2.44
2	80.89	1.01
3	80.20	0.32
4	79.93	0.05

As expected, the increase in current uncertainty (T = 0) induces a more cautious first-period response, while a more intense, though decreasing response is apparent as the increase in future uncertainty moves farther away into the future (T = 1,…, 4). Also as expected, the absolute value of the difference between the policy responses in the experiments and the

[13] Will intensity ever prevail over caution? A formal answer to this question may be, if possible, very difficult to obtain, since it amounts to show that the value of the corresponding sum of expanding expressions of the form (3.3) may become larger than the absolute value of (3.2). Numerical experimentation suggests that this possibility may not exist, unless the absolute value of a is greater than 1. In this case, the numerical value of the expressions of the form (3.3) may well become increasing as future uncertainty moves or expands into the future, inducing an increasing intensity in the first-period optimal response of u.

[14] The experiments were performed with DUALI. See Amman and Kendrick (1999).

baseline response - $\left|u_0 - u_{0(base)}\right|$ - is larger for the case of current uncertainty than for any case of future uncertainty, so that caution prevails in all cases.[15]

4. Conclusions

This article analyzed the trade-off between "caution" and "intensity" in the use of the control variable is an optimal policy problem defined as a Dynamic Stochastic Quadratic Linear Problem with a one-state one-control state equation, where the control parameter is uncertain. It showed that the trade-off between "caution" and "intensity" will always depend on the timing of the uncertainty. Given an increase in current uncertainty and an equal increase in future uncertainty, caution will always prevail over intensity. The prevalence of caution will be enlarged as the increase in future uncertainty moves farther away into the future, while it will be reduced as the increase in future uncertainty expands into the future.

[15] Notice that the sum of the series of absolute value differences for the cases of future uncertainty (T = 1, ... , 4) is much smaller than in the case of current uncertainty (T = 0), so that the prevalence of caution over intensity, though reduced, holds even in the case of expansion in future uncertainty. See footnote 13.

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